

A factorization of a super-conformal map

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Abstract

A meromorphic function, a super-conformal map, and a minimal surface are factored into a product of two maps by modeling the Euclidean four-space and the complex Euclidean plane on the set of all quaternions. One of these two maps is a holomorphic map or a meromorphic map. These conformal maps adopt properties of a holomorphic function or a meromorphic function. Analogs of the Liouville theorem, the Schwarz lemma, the Schwarz-Pick theorem, the Weierstrass factorization theorem, the Abel-Jacobi theorem, and a relation between zeros of a minimal surface and branch points of a super-conformal map are obtained.

1 Introduction

Pedit and Pinkall considered a conformal map from a Riemann surface to the Euclidean space of dimension four to be an analog of a holomorphic function or a meromorphic function in [14], by modeling \mathbb{R}^4 on the set of all quaternions \mathbb{H} . Let M be a Riemann surface with complex structure J . Given a conformal map from M to \mathbb{R}^4 , there exists a complex structure of \mathbb{R}^4 , parametrized by M , such that the conformal map is holomorphic about this complex structure at each point of M (see Figure 1).

map	equation
A conformal map $\mathfrak{f}: M \rightarrow \mathbb{H}$	$d\mathfrak{f} \circ J = N d\mathfrak{f},$ $N: M \rightarrow \text{Im } \mathbb{H}, N^2 = -1$
A holomorphic function $\mathfrak{h}: M \rightarrow \mathbb{C}$	$d\mathfrak{h} \circ J = i d\mathfrak{h}$

Table 1: A conformal map and a holomorphic function.

The motivation for making this interpretation of a conformal map is to obtain its global properties. After a long history, the theory of meromorphic functions has been successful in obtaining global properties of meromorphic functions. Compared with the theory of meromorphic functions, the theory of conformal maps seems to be insufficiently developed in obtaining global properties.

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The above interpretation raises a problem whether a conformal map adopts to a property of a meromorphic function. In [14], the order of a zero of a conformal map and the degree of a conformal map are defined (Theorem 3.2, Definition 3.2). The Riemann-Roch theorem for conformal maps is proved (Section 4). Quaternionic holomorphic curves in the quaternionic projective space in [8] succeeds various properties of meromorphic functions.

This important achievement mainly arises for a compact Riemann surface without boundary. However, studies on open Riemann surfaces are still in their infancy. We will study whether a conformal map adopts properties of a meromorphic function on an open Riemann.

Factoring a conformal map into a product of maps is an effective way to attack this problem. The example, in Example of [14], implies that a conformal map can be factored into a product of two conformal maps. The topic in [12] is considered as whether a Lagrangian conformal map becomes a product of two Lagrangian conformal maps.

Restricting the scalar of \mathbb{H} to \mathbb{C} , we identify the quaternionic vector space \mathbb{H} with a complex vector space \mathbb{C}^2 . If one of the factors of a conformal map is a meromorphic map into \mathbb{C}^2 , then the conformal map adopts a properties of a meromorphic function, such as zeros and poles, through the meromorphic factor. If the remaining factor reflects properties of a conformal map, this factorization would be a useful factorization in investigating a conformal map. In this paper, we provide this type of factorization of a meromorphic function, a super-conformal map, and a minimal surface.

Let S^2 be the two-sphere with radius one centered at the origin with respect to all imaginary parts of the quaternions $\text{Im } \mathbb{H}$. A holomorphic map and an anti-holomorphic map from a Riemann surface to $\mathbb{C}P^1$ are super-conformal maps (see Lemma 5). They are a meromorphic function or the conjugate of a meromorphic function on a Riemann surface. The following provides a factorization of a meromorphic function.

Theorem 1. *Let M be a Riemann surface.*

(a) *A map $N: M \rightarrow \mathbb{C}P^1 \cong S^2$ is a holomorphic map if and only if, for any point $p \in M$, there exist holomorphic functions λ_0 and λ_1 at p such that λ_0 and λ_1 have no common zero and that $N = (\lambda_0 + j\lambda_1)i(\lambda_0 + j\lambda_1)^{-1}$.*

(b) *A map $N: M \rightarrow \mathbb{C}P^1 \cong S^2$ is an anti-holomorphic map if and only if, for any point $p \in M$, there exist holomorphic functions λ_0 and λ_1 at p such that λ_0 and λ_1 has no common zero and that $N = -(\lambda_0 + j\lambda_1)i(\lambda_0 + j\lambda_1)^{-1}$ except common zeros of λ_0 and λ_1 .*

When M is compact without boundary and λ_0 and λ_1 are meromorphic, we show that the degree of N is equal to the degrees of λ_0 and of λ_1 (Corollary 3).

A super-conformal map is a conformal map whose curvature ellipse is a circle at each point ([2]). A holomorphic map and an anti-holomorphic map from M to $S^2 \cong \mathbb{C}P^1$ are super-conformal maps (see Lemma 5). The curvature ellipse is a central topic of the study of surfaces in \mathbb{R}^4 . There is comprehensive explanation for classical results in [20], [19], and [17]. A super-conformal map is also called a Borůvka's surface after Borůvka's study ([1]) or a Wintgen ideal surface

([16]) because the equality holds in Wintgen's inequality ([18]). A superminimal surface in \mathbb{R}^4 in [17] is a minimal super-conformal map.

Rouxel [15] showed that a conformal transform of a super-conformal map is a super-conformal map. Castro [3] showed that the Whitney sphere is the only Lagrangian super-conformal map from a closed Riemann surface. Chen [4] classified all super-conformal maps such that the absolute value of the Gaussian curvature is equal to that of the normal curvature at each point. Friedrich [17] described superminimal surfaces in terms of the twistor space. A super-conformal map in \mathbb{R}^4 is a stereographic projection of S^4 composed with the twistor projection of a holomorphic map from M to \mathbb{CP}^3 , and that a super-conformal map is a Willmore surface with vanishing Willmore energy (see [2]). In [11], [10] and [6], it is shown that a holomorphic null curve is associated with a super-conformal map.

In [2], two maps from M to $S^2 \cong \mathbb{CP}^1$ are associated with a conformal map from M to \mathbb{H} . These maps are called the left normal and the right normal of a conformal map. A super-conformal map has an anti-holomorphic left normal or an anti-holomorphic right normal. A minimal surface has a holomorphic left normal and a holomorphic right normal. For these conformal maps, we consider the case where the left normal is a non-constant holomorphic map or a non-constant anti-holomorphic map $N: M \rightarrow N(M) \subsetneq S^2$. This case implies the existence of a non-constant holomorphic function on M . Hence, M is assumed to be an open Riemann surface.

We contemplate a super-conformal map with anti-holomorphic left normal. Let S^3 be the three-sphere with radius one centered at the origin in \mathbb{H} . We model the complex projective line \mathbb{CP}^1 on S^2 . Lemma 1 in Section 4 ensures the existence of $a \in S^3$ such that $Na + ai \neq 0$ on M for any map $N: M \rightarrow N(M) \subsetneq S^2$. The following provides a factorization of a super-conformal map.

Theorem 2. *Let $N: M \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$ be an anti-holomorphic map and $a \in S^3$ such that $\psi := Na + ai$ does not vanish on M . A map $f: M \rightarrow \mathbb{H}$ is super-conformal with left normal N if and only if $f = \psi(\lambda_0 + \lambda_1 j)$ with holomorphic functions λ_0 and λ_1 on M .*

This theorem is a variant of Theorem 3.1 in [14]. We factor the Whitney sphere by this theorem. We introduce the notion of a conformal map with poles (Definition 1). A complete minimal surface of finite total curvature is a conformal map with poles (Proposition 1). We prove an analog of the Liouville theorem (Corollary 4), the Schwartz lemma (Corollary 5) and the Schwarz-Pick theorem (Corollary 6). We use the notion of a divisor of a conformal map and prove an analog of the Weierstrass factorization theorem (Corollary 8) and that of the Abel-Jacobi theorem (Corollary 9). We have a factorization of the Whitney sphere by this theorem.

We assume that M is simply connected. Because we have assumed that M is open, M is bi-holomorphic to \mathbb{C} or $\{z \in \mathbb{C} \mid |z| < 1\}$ by the uniformization theorem (see Forster [7]). Then, Theorem 1 in [11] and Theorem 1 in [10] imply the following factorization of a minimal surface, which is the real part of a holomorphic null curve.

Theorem 3. Let $N: M \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$ be a holomorphic map and $a \in S^2$ such that $\psi := -Na + ai$ does not vanish on M . For complex functions λ_0 and λ_1 on M , set $Q_{\lambda_0, \lambda_1} := \{p \in M \mid (dN)_p(\lambda_0(p) + \lambda_1(p)j) = 0\}$.

(a) If $\Phi := f + ig: M \rightarrow \mathbb{C} \otimes \mathbb{H}$ is a holomorphic null curve and f and g are minimal surfaces with left normal N , then there exist holomorphic functions λ_0 and λ_1 on M , and $\mu: M \setminus Q_{\lambda_0, \lambda_1} \rightarrow \mathbb{H}$ with $\psi d(\lambda_0 + \lambda_1 j) = dN a(\lambda_0 + \lambda_1 j)\mu$, such that $f = a(\lambda_0 + \lambda_1 j)(\mu - 1)$ and $g = -Na(\lambda_0 + \lambda_1 j)\mu + ai(\lambda_0 + \lambda_1 j)$ up to constant addition.

(b) Let λ_0 and λ_1 be holomorphic functions on M . Define a map $\mu: M \setminus Q_{\lambda_0, \lambda_1} \rightarrow \mathbb{H}$ by $\psi d(\lambda_0 + \lambda_1 j) = dN a(\lambda_0 + \lambda_1 j)\mu$. Then, the maps $f := a(\lambda_0 + \lambda_1 j)(\mu - 1)$ and $g := -Na(\lambda_0 + \lambda_1 j)\mu + ai(\lambda_0 + \lambda_1 j)$ are minimal surfaces with left normal N and $\Phi := f + ig: M \setminus Q_{\lambda_0, \lambda_1} \rightarrow \mathbb{H}$ is a holomorphic null curve.

From this theorem, we have a relation between zeros of a minimal surface and branch points of a super-conformal map (Corollary 10). We apply this relation to the Enneper surface and obtain the corresponding super-conformal map with a single branch point and a factorization of the Enneper surface.

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2 Conformal maps

We recall a conformal map from a Riemann surface to \mathbb{R}^4 ([14]) and introduce the notion of a pole and a divisor of a conformal map.

Let M be a Riemann surface with complex structure J^M . For a one-form ω on M , we define a one-form $*\omega$ on M by setting $*\omega := \omega \circ J^M$. A one-form ω with values in the set of all complex numbers \mathbb{C} is decomposed into the one-form of type $(1, 0)$ and that of type $(0, 1)$ (see Forster [7]).

We model \mathbb{R}^4 on the set of all quaternions \mathbb{H} and \mathbb{R}^3 on the set of all purely imaginary quaternions $\text{Im } \mathbb{H}$. For $a \in \mathbb{H}$, we denote by $\text{Re } a$ the real part of a and by $\text{Im } a$ the imaginary part of a . For a quaternion a , we denote by \bar{a} the quaternionic conjugate of a . Then, the inner product of a and $b \in \mathbb{H}$ is $\langle a, b \rangle := \text{Re}(\bar{a}b) = 2^{-1}(\bar{a}b + \bar{b}a)$ and the norm of $a \in \mathbb{H}$ is $|a| := (\bar{a}a)^{1/2}$. If $a, b \in \text{Im } \mathbb{H}$, then $ab = -\langle a, b \rangle + a \times b$, where \times is the cross product. Let S^2 be the sphere of radius one centered at the origin in $\text{Im } \mathbb{H}$. Then, $S^2 = \{a \in \text{Im } \mathbb{H} \mid a^2 = -1\}$. Hence, S^2 is the set of all square roots of -1 in $\text{Im } \mathbb{H}$. Let S^3 be the three-sphere with radius one centered at the origin in \mathbb{H} . For a map $N: M \rightarrow N(M) \subsetneq S^2$ and $a \in S^3$, set $\psi := Na + ai$. Then $N\psi = \psi i$.

The following lemma is used to discuss a pole of a conformal map later.

Lemma 1. For a map $N: M \rightarrow N(M) \subsetneq S^2$, there exists $a \in S^3$ such that ψ does not vanish on M .

Proof. For $a \in S^4$, the map $c \mapsto aca^{-1}$ is a Euclidean motion in $\text{Im } \mathbb{H}$. Hence, there exists $a \in S^3$ such that $-aia^{-1} \neq N(p)$ for any $p \in M$. Hence, $Na + ai$ does not vanish. \square

Fix a map $N: M \rightarrow S^2$. We use N instead of i for the decomposition of a one-form with values in \mathbb{H} . Because the multiplication in \mathbb{H} is not commutative, two kinds of a one-form with values in \mathbb{H} play a role of a one-form with values in \mathbb{C} of type $(1, 0)$ as follows.

Let ω be a one-form with values in \mathbb{H} on M . We define a one-form ω_N and a one-form ω^N by setting

$$\omega_N := \frac{1}{2}(\omega - N * \omega), \quad \omega^N := \frac{1}{2}(\omega - * \omega N).$$

Then, ω decomposes because $\omega = \omega_N + \omega_{-N} = \omega^N + \omega^{-N}$. We see that $*\omega_N = N\omega_N$ and $*\omega^N = \omega^N N$. Clearly, $\omega = \omega_N$ if and only if $\omega_{-N} = 0$. Similarly, $\omega = \omega^N$ if and only if $\omega^{-N} = 0$. The quaternionic conjugation provides an identity $\overline{\omega_N} = \overline{\omega}^{-N}$. We have the following decomposition of a two-form.

Lemma 2. *Let ω and η be one-forms with values in \mathbb{H} on M . Then*

$$\omega \wedge \eta = \omega^N \wedge \eta_{-N} + \omega^{-N} \wedge \eta_N.$$

Proof. Because $*\omega^N = \omega^N N$ and $*\omega = N\omega_N$, we have $\omega^N \wedge \eta_N = \omega^{-N} \wedge \eta_{-N} = 0$ (see [2], Proposition 16). Then,

$$\omega \wedge \eta = (\omega^N + \omega^{-N}) \wedge (\eta_N + \eta_{-N}) = \omega^N \wedge \eta_{-N} + \omega^{-N} \wedge \eta_N.$$

Hence, the lemma holds. \square

We regard \mathbb{C} as a subset $\{a_0 + a_1 i \in \mathbb{H} \mid a_0, a_1 \in \mathbb{R}\}$ of \mathbb{H} . Then, \mathbb{H} is considered as a left complex vector space $\mathbb{C} \oplus \mathbb{C}j$ or a right complex vector space $\mathbb{C} \oplus j\mathbb{C}$. Let z be the standard holomorphic coordinate of \mathbb{C} and (x, y) the real coordinate such that $z = x + yi$. Then, $*dx = -dy$. For a map $f: \mathbb{C} \rightarrow \mathbb{H}$, we have

$$\begin{aligned} 2(df)_N &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy - N * \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \\ &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy - N \left(-\frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx \right) \\ &= \left(\frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} \right) dx + N \left(\frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} \right) dy \\ &= (dx + N dy) \left(\frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} \right) = (dx)_N \left(\frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} \right), \\ 2(df)^N &= \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} N \right) (dx)^N. \end{aligned}$$

A function $h: M \rightarrow \mathbb{C}$ is holomorphic if and only if $\bar{\partial}h = (dh)_{-i} = (dh)^{-i} = 0$. Let $\lambda: M \rightarrow \mathbb{H}$ be a map. If $(d\lambda)_{-i} = 0$, then $\lambda = \lambda_0 + \lambda_1 j$ with holomorphic functions $\lambda_0: M \rightarrow \mathbb{C}$ and $\lambda_1: M \rightarrow \mathbb{C}$. If $(d\lambda)^{-i} = 0$, then $\lambda = \lambda_0 + j\lambda_1$ with holomorphic functions $\lambda_0: M \rightarrow \mathbb{C}$ and $\lambda_1: M \rightarrow \mathbb{C}$.

A non-constant map $f: M \rightarrow \mathbb{H}$ is called a conformal map with left normal $N: M \rightarrow S^2$ if $(df)_{-N} = 0$ ([14], Definition 2.1). Taking the quaternionic conjugate, we have $(d\bar{f})^N = 0$. A map $f: M \rightarrow \mathbb{H}$ is called a conformal map with right normal $N: M \rightarrow S^2$ if $(df)^N = 0$ ([2], Definition 2). A holomorphic function $h: M \rightarrow \mathbb{C}$ is a conformal map with left normal i and right normal $-i$.

Let N be a constant map. We define a complex structure J of \mathbb{H} by setting $Ja := Na$ for each $a \in \mathbb{H}$. Let $(df)_{-N} = 0$. Then, $df + J * df = df + N * df = 2(df)_{-N} = 0$. Hence f is holomorphic with respect to a complex structure J . Similarly, if a complex structure J of \mathbb{H} is defined by setting $Ja := -aN$ for each $a \in \mathbb{H}$ and $(df)_N = 0$, then f is holomorphic with respect to a complex structure J .

We recall a zero of a conformal map ([14]). Let U be a coordinate neighborhood of M , $p \in U$ and z a holomorphic coordinate on U centered at p . A map $f: U \rightarrow \mathbb{H}$ vanishes to order at least $n \geq 0$ at p if $|f(z)| \leq C|z|^n$ for some constant $C > 0$. If f vanishes to order at least $n \geq 0$ at p , but f does not vanish to order at least $n+1$ at p , then a map f vanishes to order n at p . The order n depends only on f .

We assume that $f: U \rightarrow \mathbb{H}$ is conformal with left normal N such that $N(U) \subsetneq S^2$. By Lemma 1, Theorem 3.2 in [14] and Lemma 3.9 in [8], there exist a nowhere-vanishing map $\psi: U \rightarrow \mathbb{H}$ with $N\psi = \psi i$, a nowhere-vanishing map $\phi_f: U \rightarrow \mathbb{H}$, and a map $\xi: U \rightarrow \mathbb{H}$ such that

$$f(z) = \psi(z)(z^n \phi_f(z) + \xi(z)), \quad \lim_{z \rightarrow 0} \frac{|\xi(z)|}{|z|^{n+1}} < \infty.$$

The point p is called a zero of f . The integer n is called the order of f at p and denoted by $\text{ord}_p f$.

As an analog of a zero of a conformal map, we introduce the notion of a pole of a conformal map.

Lemma 3. *Let $f: U \setminus \{p\} \rightarrow \mathbb{H}$ be a conformal map with left normal N such that $N(U) \subsetneq S^2$. We assume that $|z|^n |f(z)| \leq C$ for some constant $C > 0$, but there does not exist a constant $\tilde{C} > 0$ such that $|z|^{n-1} |f(z)| \leq \tilde{C}$ for a positive integer n . Then there exists a nowhere-vanishing map $\phi_f: U \rightarrow \mathbb{H}$ and a map $\xi: U \rightarrow \mathbb{H}$ such that*

$$f(z) = \psi(z) (z^{-n} \phi_f(z) + \xi(z)), \quad \lim_{z \rightarrow 0} \frac{|\xi(z)|}{|z|^{-n+1}} < \infty. \quad (1)$$

Proof. We give a proof which is parallel to the proof of Lemma 3.9 in [8]. Because $|z|^n |f(z)| \leq C$, the map $z^{-n} \psi^{-1}(z) f(z)$ is defined on U . Let us define $\lambda := \lambda_0 + \lambda_1 j$ with $\lambda_0, \lambda_1: U \rightarrow \mathbb{C}$ by $f(z) = \psi(z) z^{-n} \lambda(z)$. Then, the equation $(df)_{-N} = 0$ becomes

$$(d\psi)_{-N} z^{-n} \lambda(z) + \psi(z) z^{-n} (d\lambda)_{-i} = 0. \quad (2)$$

Let α_0 and α_1 be complex one-forms on U such that $(d\psi)_{-N} = \psi(\alpha_0 + \alpha_1 j)$. Because

$$*(d\psi)_{-N} = -N(d\psi)_{-N} = -N\psi(\alpha_0 + \alpha_1 j)$$

$$= \psi(-i)(\alpha_0 + \alpha_1 j) = \psi(*\alpha_0 + *\alpha_1 j),$$

the one-forms α_0 and α_1 are type $(0, 1)$. The equation (2) becomes

$$\begin{aligned}\alpha_0 z^{-n} \lambda_0(z) - \alpha_1 \bar{z}^{-n} \overline{\lambda_1(z)} + z^{-n} \bar{\partial} \lambda_0 &= 0, \\ \alpha_0 z^{-n} \lambda_1(z) + \alpha_1 \bar{z}^{-n} \overline{\lambda_0(z)} + z^{-n} \bar{\partial} \lambda_1 &= 0.\end{aligned}$$

Simplifying this system of equations, we have

$$\begin{aligned}\alpha_0 \lambda_0(z) - \alpha_1 \left(\frac{z}{\bar{z}}\right)^n \overline{\lambda_1(z)} + \bar{\partial} \lambda_0 &= 0, \\ \alpha_0 \lambda_1(z) + \alpha_1 \left(\frac{z}{\bar{z}}\right)^n \overline{\lambda_0(z)} + \bar{\partial} \lambda_1 &= 0.\end{aligned}$$

Hence

$$\begin{pmatrix} \bar{\partial} \lambda_0 \\ \bar{\partial} \lambda_1 \end{pmatrix} + \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} \begin{pmatrix} \lambda_0(z) \\ \lambda_1(z) \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_1 (z/\bar{z})^n \\ \alpha_1 (z/\bar{z})^n & 0 \end{pmatrix} \begin{pmatrix} \overline{\lambda_0(z)} \\ \overline{\lambda_1(z)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system of equations has the same form as equation (51) in [8]. Hence, there exists a nowhere-vanishing map $\phi_f: U \rightarrow \mathbb{H}$ and a map $\xi: U \rightarrow \mathbb{H}$ such that (1) holds for a positive integer n . \square

Definition 1. We call a point p in Lemma 3 a pole of f and the integer n the order of f at a pole p . For a Riemann surface M which is biholomorphic to a Riemann surface \tilde{M} with discrete set \mathcal{P} removed, we call $f: M \rightarrow \mathbb{H}$ a conformal map with poles at \mathcal{P} if each point in \mathcal{P} is a pole of f .

If f is a meromorphic function on U , then $N = i$. Choosing $a = -i/2$, we have $\psi = 1$. Then, the order of f as a conformal map is equal to that as a meromorphic function.

Recall that a divisor on M is a map $D: M \rightarrow \mathbb{Z}$ such that, for any compact subset K of M , the set $\{p \in M \mid D(p) \neq 0\} \cap K$ is a finite set (see [7]). The set $\{p \in M \mid D(p) \neq 0\}$ is called the support of D and denoted by $\text{supp } D$. The degree of a divisor D is defined by $\deg D := \sum_{p \in M} D(p)$. We denote by $\text{Div}(M)$ the set of all divisors on M .

We introduce the notion of the divisor of a conformal map as follows. Let \mathcal{P} be a subset of M such that, for any compact subset K of M , the set $\mathcal{P} \cap K$ is a finite set. Let $f: M \setminus \mathcal{P} \rightarrow \mathbb{H}$ be a conformal map with left normal N and poles at \mathcal{P} . We define $\text{ord}_p f$ by

$$\text{ord}_p f = \begin{cases} 0, & \text{if } f \text{ is neither zero nor pole at } p, \\ k, & \text{if } f \text{ has a zero of order } k \text{ at } p, \\ -k, & \text{if } f \text{ has a pole of order } k \text{ at } p, \\ \infty, & \text{if } f \text{ is identically zero in a neighborhood of } p. \end{cases}$$

We define a map $(f): M \rightarrow \mathbb{Z}$ by $(f)(p) := \text{ord}_p f$ for each $p \in M$. Let us define nonnegative maps $Z: M \rightarrow \mathbb{Z}$ and $P: M \rightarrow \mathbb{Z}$ by $(f) = Z - P$. The map P is a divisor on M by the assumption. The map (f) is a divisor on M if and only if Z is a divisor on M .

Definition 2. We assume that (f) is a divisor on M . We call (f) the divisor of f and the map f a conformal map with divisor (f) . We call the divisors Z and P the zero divisor of f and the polar divisor of f respectively.

There exists an important class of conformal maps with poles.

Proposition 1. *Let M be an open Riemann surface and $f: M \rightarrow \mathbb{H}$ a conformal map which is a complete minimal surface of finite total curvature with respect to the induced (singular) metric. Then f is a conformal map with poles.*

Proof. By Chern and Osserman [5] and Moriya [13], M is biholomorphic to a closed Riemann surface with a set of a finite number of points $\mathcal{P} = \{p_1, \dots, p_r\}$ removed. Let $f_m: M \rightarrow \mathbb{R}$ ($m = 0, 1, 2, 3$), $f = f_0 + f_1i + f_2j + f_3k$. At each point $p_l \in \mathcal{P}$, there exists a meromorphic function $F_{m,l}$ at p_l such that $\operatorname{Re} F_{m,l} = f_m$ ($m = 0, 1, 2, 3$, $l = 1, \dots, r$). Let z be a holomorphic coordinate z centered at $p_l \in \mathcal{P}$. Then, there exists a positive integer $n_{m,l}$ such that $|z|^{n_{m,l}}|F_{m,l}(z)| \leq C_{m,l}$ for some constant $C_{m,l} > 0$, but there does not exist a constant $\tilde{C}_{m,l} > 0$ such that $|z|^{n_{m,l}-1}|F_{m,l}(z)| \leq \tilde{C}_{m,l}$. Because $|f_m(z)| \leq |F_{m,l}(z)|$, we have $|z|^{n_{m,l}}|f_m(z)| \leq C_{m,l}$ for some constant $C_{m,l} > 0$. Let $n_l = \min\{n_{m,l} \mid m = 0, 1, 2, 3\}$. Then

$$|z|^{n_l}|f(z)| \leq |z|^{n_l} \sum_{m=0}^3 |f_m(z)| \leq |z|^{n_l} \sum_{m=0}^3 |z|^{n_{m,l}-n_l} C_{m,l}.$$

Hence $|z|^{n_l}|f(z)| \leq C_l$ for some constant $C_l > 0$.

Let $n_{m',l} = n_l$. Because the real part of a meromorphic function $F_{m',l}$ is $f_{m'}$, there does not exist a constant $\tilde{C}_{m',l} > 0$ such that $|z|^{n_{m',l}-1}|f_{m',l}(z)| \leq \tilde{C}_{m',l}$. Because $|f| \geq |f_{m',l}|$, there does not exist a constant $\tilde{C}_{ml} > 0$ such that $|z|^{n_{m',l}-1}|f(z)| \leq \tilde{C}_l$. Hence p_l is a pole of f . Then f is a conformal map with poles. \square

From the proof of Proposition 1, we have the following corollary immediately.

Corollary 1. *Let \tilde{M} be a closed Riemann surface, $f = f_0 + f_1i + f_2j + f_3k: \tilde{M} \setminus \{p_1, \dots, p_r\} \rightarrow \mathbb{H}$ be a complete minimal surface of finite total curvature and $F_{m,l}$ a meromorphic function at p_l such that $\operatorname{Re} F_{m,l} = f_m$ ($m = 0, 1, 2, 3$, $l = 1, \dots, r$). Then $\operatorname{ord}_{p_l} f = \min\{\operatorname{ord}_{p_l} F_{m,l} \mid m = 0, 1, 2, 3\}$ ($l = 1, \dots, r$).*

3 Meromorphic functions

We factor a meromorphic function on a Riemann surface.

The map $\operatorname{st}: S^2 \setminus \{k\} \rightarrow \mathbb{C}$ defined by

$$\operatorname{st}(x_1i + x_2j + x_3k) = \frac{x_1}{1-x_3} + \frac{x_2}{1-x_3}i \quad (x_1, x_2, x_3 \in \mathbb{R})$$

is the stereographic projection from k . We model S^2 on the complex projective line $\mathbb{C}P^1$ so that st is a holomorphic map.

Let $N: M \rightarrow S^2$ be a conformal map. Then $N: M \rightarrow S^2 \cong \mathbb{CP}^1$ is holomorphic or anti-holomorphic. Differentiating the equation $N^2 = -1$, we have $dN N + N dN = 0$. By Lemma 2 in [2], we have $(dN)_N = 0$ or $(dN)_{-N} = 0$.

Lemma 4. *A map $N: M \rightarrow S^2 \cong \mathbb{CP}^1$ is holomorphic if and only if N satisfies $(dN)_N = (dN)^{-N} = 0$.*

Proof. For a map $N: M \rightarrow S^2$, we define real-valued functions n_1, n_2 , and n_3 by $N = n_1 i + n_2 j + n_3 k$. Put $M_+ := M \setminus \{p \in M \mid N(p) = k\}$ and $M_- := M \setminus \{p \in M \mid N(p) = -k\}$.

We assume that $(dN)_N = (dN)^{-N} = 0$. This equation becomes

$$\begin{aligned} n_1 * dn_1 + n_2 * dn_2 - n_3 * dn_3 &= 0, \\ dn_1 - n_2 * dn_3 + n_3 * dn_2 &= 0, \\ dn_2 - n_3 * dn_1 + n_1 * dn_3 &= 0, \\ dn_3 - n_1 * dn_2 + n_2 * dn_1 &= 0. \end{aligned}$$

We define functions l_1 and l_2 with values in \mathbb{R} on M_+ by

$$l_1 + l_2 i := \text{st} \circ N = \frac{n_1}{1 - n_3} + \frac{n_2}{1 - n_3} i.$$

Then,

$$\begin{aligned} dl_1 &= \frac{dn_1(1 - n_3) - n_1 d(1 - n_3)}{(1 - n_3)^2} = \frac{dn_1 - dn_1 n_3 + n_1 dn_3}{(1 - n_3)^2} = \frac{dn_1 + * dn_2}{(1 - n_3)^2}, \\ dl_2 &= \frac{dn_2 - dn_2 n_3 + n_2 dn_3}{(1 - n_3)^2} = \frac{dn_2 - * dn_1}{(1 - n_3)^2}. \end{aligned}$$

Hence,

$$(d(l_1 + l_2 i))_{-i} = 0.$$

Then, $l_1 + l_2 i$ is a holomorphic function. Hence, $N|_{M_+}$ is a holomorphic map.

The map

$$-iNi = n_1 i - n_2 j - n_3 k$$

is a rotation of N centered at the origin. We have $M \setminus \{p \in M \mid -i(N(p))i = k\} = M_-$. In an analogous discussion as above, we show that $-iNi|_{M_-}$ is a holomorphic map. This is equivalent to having $N|_{M_-}$ as a holomorphic map. Therefore, N is a holomorphic map on M .

Conversely, we assume that N is holomorphic. Then, $(d(l_1 + l_2 i))_{-i} = 0$. Because

$$N = \frac{2l_1}{l_1^2 + l_2^2 + 1} i + \frac{2l_2}{l_1^2 + l_2^2 + 1} j + \frac{l_1^2 + l_2^2 - 1}{l_1^2 + l_2^2 + 1} k,$$

we have

$$\begin{aligned}
dN &= \left(\frac{2(-l_1^2 + l_2^2 + 1)}{(l_1^2 + l_2^2 + 1)^2} dl_1 - \frac{4l_1 l_2}{(l_1^2 + l_2^2 + 1)^2} dl_2 \right) i \\
&\quad + \left(\frac{2(l_1^2 - l_2^2 + 1)}{(l_1^2 + l_2^2 + 1)^2} dl_1 - \frac{4l_1 l_2}{(l_1^2 + l_2^2 + 1)^2} dl_2 \right) j \\
&\quad + \left(\frac{4l_1}{(l_1^2 + l_2^2 + 1)^2} dl_1 + \frac{4l_2}{(l_1^2 + l_2^2 + 1)^2} dl_2 \right) k, \\
N dN &= \left(\frac{4l_1 l_2}{(l_1^2 + l_2^2 + 1)^2} dl_1 + \frac{2(-l_1^2 + l_2^2 + 1)}{(l_1^2 + l_2^2 + 1)^2} dl_2 \right) i \\
&\quad + \left(\frac{4l_1 l_2}{(l_1^2 + l_2^2 + 1)^2} dl_1 + \frac{2(l_1^2 - l_2^2 + 1)}{(l_1^2 + l_2^2 + 1)^2} dl_2 \right) j \\
&\quad + \left(-\frac{4l_2}{(l_1^2 + l_2^2 + 1)^2} dl_1 + \frac{4l_1}{(l_1^2 + l_2^2 + 1)^2} dl_2 \right) k.
\end{aligned}$$

Hence, $(dN)_N|_{M_+} = 0$. A similar discussion for $-iNi$ shows that $(dN)_N|_{M_-} = 0$. Hence, $(dN)_N = (dN)^{-N} = 0$ over M . \square

Corollary 2. *A map $N: M \rightarrow S^2 \cong \mathbb{C}P^1$ is anti-holomorphic if and only if $-N$ is holomorphic. In other words, a map $N: M \rightarrow S^2$ is anti-holomorphic if and only if $(dN)_{-N} = (dN)^N = 0$.*

Proof. We have

$$\begin{aligned}
\text{st} \circ (-N) &= \frac{-(n_1 + n_2 i)}{1 + n_3} = \frac{-(n_1^2 + n_2^2)}{(1 + n_3)(n_1 - n_2 i)} \\
&= \frac{-(1 - n_3^2)}{(1 + n_3)(n_1 - n_2 i)} = \frac{-(1 - n_3)}{n_1 - n_2 i} = -\frac{1}{\text{st} \circ N}.
\end{aligned}$$

Hence, N is anti-holomorphic if and only if $-N$ is holomorphic; that is $(dN)_{-N} = (dN)^N = 0$. \square

Proof of Theorem 1. (b) follows from (a) and Corollary 2. We show (a).

We assume that λ_0 and λ_1 are holomorphic functions at p without common zero. Set $\lambda := \lambda_0 + j\lambda_1$. Then $(d\lambda)^{-i} = 0$. We have

$$\begin{aligned}
dN &= d\lambda i\lambda^{-1} - \lambda i\lambda^{-1} d\lambda \lambda^{-1}, \\
N * dN &= \lambda i\lambda^{-1} * d\lambda i\lambda^{-1} + * d\lambda \lambda^{-1} = -\lambda i\lambda^{-1} d\lambda \lambda^{-1} + d\lambda i\lambda^{-1}.
\end{aligned}$$

Hence, $(dN)_N = 0$. Then, N is holomorphic at p by Lemma 4.

Conversely, we assume that N is holomorphic at p . Then, $(dN)_N = 0$. For any $a \in S^3$, the quaternion aia^{-1} is a rotation of i centered at the origin in $\text{Im } \mathbb{H}$. Hence, there exists a map ξ with $|\xi| = 1$ such that $N = \xi i \xi^{-1}$.

The equation $(dN)_N = 0$ becomes

$$d\xi i \xi^{-1} - \xi i \xi^{-1} d\xi \xi^{-1} = \xi i \xi^{-1} * d\xi i \xi^{-1} + * d\xi \xi^{-1}.$$

Simplifying this equation, we have

$$i\xi^{-1}(d\xi)^{-i} = \xi^{-1}(d\xi)^{-i}i.$$

Hence $\omega := \xi^{-1}(d\xi)^{-i}$ is a complex $(0,1)$ -form. Let ξ_0 and ξ_1 be complex functions such that $\xi = \xi_0 + j\xi_1$. Because $|\xi| = 1$, the functions ξ_0 and ξ_1 have no common zero. Then,

$$\bar{\partial}\xi_0 + j\bar{\partial}\xi_1 = \xi_0\omega + j\xi_1\omega.$$

Hence,

$$\omega = \bar{\partial}\log\xi_0 = \bar{\partial}\log\xi_1.$$

Then, there exist holomorphic functions λ_0 and λ_1 at p without common zero such that $\lambda_0\xi_0 = \lambda_1\xi_1$. Then

$$\begin{aligned} N &= \xi i \xi^{-1} = (\xi_0 + j\xi_1)i(\xi_0 + j\xi_1)^{-1} \\ &= (\lambda_0\xi_0 + j\lambda_0\xi_1)\lambda_0^{-1}i\lambda_0(\lambda_0\xi_0 + j\lambda_0\xi_1)^{-1} \\ &= (\lambda_1\xi_1 + j\lambda_0\xi_1)i(\lambda_1\xi_1 + j\lambda_0\xi_1)^{-1} \\ &= (\lambda_1 + j\lambda_0)i(\lambda_1 + j\lambda_0)^{-1}. \end{aligned}$$

Thus, the theorem holds. \square

If $N = (\lambda_0 + j\lambda_1)i(\lambda_0 + j\lambda_1)^{-1}$ is a holomorphic map with holomorphic functions λ_0 and λ_1 , then $N = \Lambda i \bar{\Lambda}$ with $\Lambda := (\lambda_0 + j\lambda_1)/|\lambda|$.

If M is a closed Riemann surface and λ_0 and λ_1 are meromorphic on M , then $N: M \rightarrow S^2 \cong \mathbb{CP}^1$ is a holomorphic map. From this factorization, we have a relation between the degree of N and the degree of λ when M is closed.

Corollary 3. *Let M be a closed Riemann surface, λ_0 and λ_1 are meromorphic functions on M , $\lambda := \lambda_0 + j\lambda_1$, and $N := \lambda i \lambda^{-1}$. The degree of a holomorphic map N is m if and only if the degree of λ_0 and λ_1 are m .*

Proof. We assume that the degree of N is m . The equation $N = i$ has m solutions counting multiplicities. This equation is equivalent to the equation $i\lambda = \lambda i$. Rewriting this equation, we have $\lambda_0 i = \lambda_0 i$ and $-\lambda_1 i = \lambda_1 i$. The former equation is trivial; the latter is equivalent to $\lambda_1 = 0$. Hence the equation $\lambda_1 = 0$ has m solutions counting multiplicities. Then, λ_1 is a meromorphic function of degree m . Next, we consider the equation $N = -i$. This equation is equivalent to $\lambda_0 = 0$. Hence λ_0 is a meromorphic function of degree m . The converse is trivial. \square

4 Super-conformal maps

We factor a super-conformal map.

We recall the definition and basic properties of a super-conformal map (see [2]). A conformal map $f: M \rightarrow \mathbb{H}$ is called a super-conformal map if its curvature ellipse is a circle. A conformal map f is super-conformal if and only if its left normal or its right normal is anti-holomorphic. Let $N: M \rightarrow S^2$ be the left normal of f and $R: M \rightarrow S^2$ the right normal of f . Then f is super-conformal if and only if $(dN)_{-N} = 0$ or $(dR)_{-R} = 0$ by Corollary 2.

Lemma 5. *A holomorphic map and an anti-holomorphic map from M to $\mathbb{CP}^1 \cong S^2$ are super-conformal.*

Proof. Let $N: M \rightarrow S^2$ be a holomorphic map. By Lemma 4, $(dN)_N = 0$. Then, N is a conformal map with left normal $-N$. The map $-N$ is anti-holomorphic by Corollary 2. Hence, N is super-conformal.

Let $N: M \rightarrow \mathbb{CP}^1 \cong S^2$ be an anti-holomorphic map. By Corollary 2, $(dN)_{-N} = 0$. Then, N is a conformal map with left normal N . Hence, N is super-conformal. \square

It is known that a super-conformal map is a stereographic projection composed with the twistor projection of a holomorphic map from a Riemann surface to \mathbb{CP}^3 ([2], Theorem 5). Hence, for holomorphic functions $\lambda_0, \lambda_1, \lambda_2$ and λ_3 , a map

$$f = (\lambda_0 + \lambda_1 j)^{-1} (\lambda_2 + \lambda_3 j) \quad (3)$$

is a super-conformal map with anti-holomorphic left normal. Indeed,

$$\begin{aligned} df &= -(\lambda_0 + \lambda_1 j)^{-1} (d\lambda_0 + d\lambda_1 j) (\lambda_0 + \lambda_1 j)^{-1} (\lambda_2 + \lambda_3 j) \\ &\quad + (\lambda_0 + \lambda_1 j)^{-1} (d\lambda_2 + d\lambda_3 j), \\ *df &= (\lambda_0 + \lambda_1 j)^{-1} i (\lambda_0 + \lambda_1 j) df. \end{aligned}$$

Hence f is conformal with left normal $(\lambda_0 + \lambda_1 j)^{-1} i (\lambda_0 + \lambda_1 j)$. We have

$$\begin{aligned} (\lambda_0 + \lambda_1 j)^{-1} i (\lambda_0 + \lambda_1 j) &= (\overline{\lambda_0} - \lambda_1 j) i (\overline{\lambda_0} - \lambda_1 j)^{-1} \\ &= (-\overline{\lambda_0} j - \lambda_1) j i (-j) (-\overline{\lambda_0} j - \lambda_1)^{-1} \\ &= -(\lambda_1 + j\lambda_0) i (\lambda_1 + j\lambda_0)^{-1}. \end{aligned}$$

By Theorem 1, the map $-(\lambda_1 + j\lambda_0) i (\lambda_1 + j\lambda_0)^{-1}$ is an anti-holomorphic map. Hence f is a super-conformal map with anti-holomorphic left normal. We can consider (3) as a factorization of a super-conformal map. Conversely, if an anti-holomorphic map $-(\lambda_1 + j\lambda_0) i (\lambda_1 + j\lambda_0)^{-1}$ is given, then a map $f = (\lambda_0 + \lambda_1 j)^{-1} (\lambda_2 + \lambda_3 j)$ with holomorphic functions λ_2 and λ_3 is a super-conformal map.

Theorem 2 provides another factorization of a super-conformal map.

Proof of Theorem 2. Because ψ is nowhere-vanishing, any map $f: M \rightarrow \mathbb{H}$ is factored by the product $\psi(\lambda_0 + j\lambda_1)$ with complex functions λ_0 and λ_1 on M .

Let $\lambda := \lambda_0 + \lambda_1 j$. The functions λ_0 and λ_1 are holomorphic if and only if $(d\lambda)_{-i} = 0$. We have

$$\begin{aligned} d(\psi\lambda) &= dN a\lambda + \psi d\lambda, \\ 2(d(\psi\lambda))_{-N} &= dN a\lambda + \psi d\lambda + N * (dN a\lambda + \psi d\lambda) = 2\psi(d\lambda)_{-i}. \end{aligned}$$

Hence, $f = \psi\lambda$ with $(d\lambda)_{-i} = 0$ if and only if f is super-conformal with left normal N . \square

By the above theorem, the set of super-conformal maps from M to \mathbb{H} with anti-holomorphic left normal $N: M \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$ is parametrized by two holomorphic functions. Hence, a super-conformal map adopts properties of a holomorphic function.

The following is an analog of the Liouville theorem (see [9]).

Corollary 4. *Let $f: \mathbb{C} \rightarrow \mathbb{H}$ be a super-conformal map with anti-holomorphic left normal $N: \mathbb{C} \rightarrow N(\mathbb{C}) \subsetneq S^2 \cong \mathbb{CP}^1$ and $f = \psi(\lambda_0 + \lambda_1 j)$ is the factorization by Theorem 2. We assume that $|\psi| \geq c$ for a positive number c and f is bounded. Then, f is constant.*

Proof. Because $-N: \mathbb{C} \rightarrow S^2 \cong \mathbb{CP}^1$ is holomorphic map and $|\psi| \geq c$ for a positive number c , the map N is constant by the Liouville theorem (see [9]). Then, ψ is constant and

$$|\lambda_0 + \lambda_1 j| = (|\lambda_0|^2 + |\lambda_1|^2)^{1/2} = \frac{|f|}{|\psi|}.$$

Because $|f|$ is bounded, holomorphic functions λ_0 and λ_1 are bounded entire functions. By the Liouville theorem, λ_0 and λ_1 are constant. Hence f is constant. \square

Let z be the standard holomorphic coordinate of \mathbb{C} , (x, y) the real coordinate such that $z = x + yi$ and $D := \{z \in \mathbb{C} \mid |z| < 1\}$. The following is an analog of the Schwarz lemma (see [9]).

Corollary 5. *Let $f: D \rightarrow \mathbb{H}$ be a super-conformal map with anti-holomorphic left normal $N: D \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$. We assume that $|\psi| \geq c$ for a positive number c , $f(0) = 0$ and f is bounded. Then, there exists a constant $C > 0$ such that*

$$|f(z)| \leq 2C|z|, \quad \left| \frac{\partial f}{\partial x}(0) - N(0) \frac{\partial f}{\partial y}(0) \right| \leq 2C.$$

The equality holds if and only if $N = aia^{-1}$, there exists $z_0 \in D \setminus \{0\}$ such that $|\lambda_n(z_0)| = C_n|z_0|$ ($n = 0, 1$) and $C = (C_0^2 + C_1^2)^{1/2}$.

Proof. Let $f = \psi(\lambda_0 + \lambda_1 j)$ be the factorization by Theorem 2. Because $f(0) = 0$ and ψ is nowhere-vanishing, we have $\lambda_0(0) = \lambda_1(0) = 0$. Also, because $\psi \geq c$ for a positive number c and $|f|$ is bounded, the map $\lambda := \lambda_0 + \lambda_1 j$ is bounded.

Because $|\psi| = |Na + ai| \leq |Na| + |ai| = 2$, the equality $|\psi| = 2$ holds if and only if $Na = ai$. Then N is a constant map. Because $|\lambda_n| \leq |\lambda|$ ($n = 0, 1$), the functions λ_0 and λ_1 are bounded.

Let $|\lambda_n(z)| \leq C_n$ ($n = 0, 1$). By the Schwarz lemma, we have $|\lambda_n(0)| \leq C_n|z|$ and $|(\partial\lambda_n/\partial z)(0)| \leq C_n$. There exists a point $z_0 \in D \setminus \{0\}$ such that $|\lambda_n(z_0)| = C_n|z_0|$ if and only if $\lambda_n(z) = C_n e^{\theta_n i} z$ with real-valued function θ_n . Then

$$\begin{aligned} |f(z)| &= |\psi(z)||\lambda(z)| \leq 2(|\lambda_0(z)|^2 + |\lambda_1(z)|^2)^{1/2} \leq 2(C_0^2 + C_1^2)^{1/2}|z|, \\ &\quad \left| \frac{\partial f}{\partial x}(0) - N(0) \frac{\partial f}{\partial y}(0) \right| \\ &= \left| \left(\frac{\partial \psi}{\partial x}(0) - N(0) \frac{\partial \psi}{\partial y}(0) \right) \lambda(0) + \psi(0) \left(\frac{\partial \lambda}{\partial x}(0) - i \frac{\partial \lambda}{\partial y}(0) \right) \right| \\ &= \left| \psi(0) \left(\frac{\partial \lambda}{\partial x}(0) - i \frac{\partial \lambda}{\partial y}(0) \right) \right| = |\psi(0)| \left| \frac{\partial \lambda_0}{\partial z}(0) + \frac{\partial \lambda_1}{\partial z}(0)j \right| \\ &\leq 2 \left(\left| \frac{\partial \lambda_0}{\partial z}(0) \right|^2 + \left| \frac{\partial \lambda_1}{\partial z}(0) \right|^2 \right)^{1/2} \leq 2(C_0^2 + C_1^2)^{1/2}. \end{aligned}$$

The equality holds if and only if $N = aia^{-1}$ and there exists $z_0 \in D \setminus \{0\}$ such that $|\lambda_n(z_0)| = C_n|z_0|$ ($n = 0, 1$). \square

The following is an analog of the Schwarz-Pick theorem (see [9]). Let $B^3 := \{a \in \mathbb{H} \mid |a| < 1\} \subset \mathbb{H}$. For $z_1 \in D$ and $a_1 \in B^3$, define Möbius transforms $\tau^{z_1}: D \rightarrow D$ and $\Theta^{a_1}: B^3 \rightarrow B^3$ by

$$\tau^{z_1}(z) = \frac{z - z_1}{1 - \overline{z_1}z}, \quad \Theta^{a_1}(a) = (a - a_1)(1 - \overline{a_1}a)^{-1}.$$

Let $f: D \rightarrow B^3$ be a super-conformal map with anti-holomorphic left normal. For a given $z_1 \in D$, we define a conformal map $g^{z_1}: D \rightarrow B^3$ by $g^{z_1} = \Theta^{f(z_1)} \circ f \circ (\tau^{z_1})^{-1}: D \rightarrow B^3$. Let $N_{g^{z_1}}$ be the left normal of g^{z_1} . Let P^f be the set of all $z \in D$ such that (i) $N_{g^z}(M) \subsetneq S^2$ and (ii) there exist $a \in S^3$ and a positive number c such that $|N_{g^z}a + ai| \geq c$.

Corollary 6. *Let $f: D \rightarrow B^3$ be a super-conformal map with anti-holomorphic left normal N . We assume that $z_1 \in P^f$. Then, there exists a constant $C^{z_1} > 0$ such that*

$$\frac{\left| \frac{\partial f}{\partial x}(z_1) \right|}{1 - |f(z_1)|^2} = \frac{\left| \frac{\partial f}{\partial y}(z_1) \right|}{1 - |f(z_1)|^2} \leq \frac{2C^{z_1}}{1 - |z_1|^2}.$$

We have

$$\frac{|f(z_1) - f(z)|}{|1 - \overline{f(z_1)}f(z)|} \leq 2C^{z_1} \left| \frac{z_1 - z}{1 - \overline{z_1}z} \right|$$

for all $z \in D$.

Proof. Rouxel [15] showed that a conformal transform of a super-conformal map is a super-conformal map. Hence g^{z_1} is a super-conformal map with $g^{z_1}(0) = 0$. By Corollary 5, there exists $C^{z_1} > 0$ such that $|g^{z_1}(z)| \leq 2C^{z_1} |z|$. Hence

$$\left| (f(z) - f(z_1)) \left(1 - \overline{f(z_1)} f(z) \right)^{-1} \right| \leq 2C^{z_1} \left| \frac{z - z_1}{1 - \overline{z_1} z} \right|.$$

Then

$$\frac{|f(z) - f(z_1)|}{\left| 1 - \overline{f(z_1)} f(z) \right|} \leq 2C^{z_1} \left| \frac{z - z_1}{1 - \overline{z_1} z} \right|$$

for any $z \in D$.

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_1 i$, $(x_1, x_2, y_1 \in \mathbb{R})$. Then

$$\frac{|f(x_2 + y_1 i) - f(x_1 + y_1 i)|}{\left| 1 - \overline{f(x_1 + y_1 i)} f(x_2 + y_1 i) \right|} \leq 2C^{z_1} \left| \frac{x_2 - x_1}{1 - \overline{(x_1 + y_1 i)}(x_2 + y_1 i)} \right|.$$

Hence

$$\frac{|f(x_2 + y_1 i) - f(x_1 + y_1 i)|}{|x_2 - x_1| \left| 1 - \overline{f(x_1 + y_1 i)} f(x_2 + y_1 i) \right|} \leq 2C^{z_1} \left| \frac{1}{1 - \overline{(x_1 + y_1 i)}(x_2 + y_1 i)} \right|.$$

Let x_2 tend to x_1 . Then

$$\frac{\left| \frac{\partial f}{\partial x}(z_1) \right|}{1 - |f(z_1)|^2} \leq \frac{2C^{z_1}}{1 - |z_1|^2}.$$

Because f is conformal, we have

$$\left| \frac{\partial f}{\partial x}(z_1) \right| = \left| \frac{\partial f}{\partial y}(z_1) \right|.$$

Then the corollary holds. \square

Let ds^2 be the Poincaré metric on D with curvature -1 and $ds_{B^3}^2$ be the Poincaré metric on B^3 with curvature -1 . For the standard coordinate (x, y) of \mathbb{R}^2 and the standard coordinate (a_0, a_1, a_2, a_3) of \mathbb{R}^4 , we have

$$ds_D^2 = \frac{4}{(1 - (x^2 + y^2))^2} (dx \otimes dx + dy \otimes dy),$$

$$ds_{B^3}^2 = \frac{4}{(1 - \sum_{n=0}^3 a_n^2)^2} \sum_{n=0}^3 (da_n \otimes da_n).$$

The following is a geometric interpretation of Corollary 6.

Corollary 7. *Let $f: D \rightarrow \mathbb{H}$ be a super-conformal map with anti-holomorphic left normal N such that $f(M) \subset B^3$. Then, at each point z on P^f , there exists a constant $C^z > 0$ such that $f^* ds_{B^3}^2 \leq 4(C^z)^2 ds_D^2$.*

Proof. Let f_0, f_1, f_2 and f_3 be the real-valued functions such that $f = f_0 + f_1 i + f_2 j + f_3 k$. Then,

$$\begin{aligned} f^* ds_{B^3}^2 &= \frac{4}{(1 - \sum_{n=0}^3 (f_n(z))^2)^2} \\ &\times \sum_{n=0}^3 \left(\left(\frac{\partial f_n}{\partial x}(z) \right)^2 dx \otimes dx + \left(\frac{\partial f_n}{\partial y}(z) \right)^2 dy \otimes dy \right) \\ &= \frac{4}{(1 - |f(z)|^2)^2} \left(\left| \frac{\partial f}{\partial x}(z) \right|^2 dx \otimes dx + \left| \frac{\partial f}{\partial y}(z) \right|^2 dy \otimes dy \right). \end{aligned}$$

By Corollary 6, there exists $C^z > 0$ such that

$$\frac{4}{(1 - |f(z)|^2)^2} \left| \frac{\partial f}{\partial x}(z) \right|^2 = \frac{4}{(1 - |f(z)|^2)^2} \left| \frac{\partial f}{\partial y}(z) \right|^2 \leq \frac{16(C^z)^2}{(1 - |z|^2)^2}$$

at each $z \in P^f$. Hence

$$f^* ds_{B^3}^2 \leq \frac{16(C^z)^2}{(1 - |z|^2)^2} (dx \otimes dx + dy \otimes dy) = 4(C^z)^2 ds_D^2$$

at each $z \in P^f$. □

$$\begin{array}{ccc} (D, f^* \text{Re}\langle \cdot, \cdot \rangle) & \xrightarrow{f} & (\mathbb{H}, \text{Re}\langle \cdot, \cdot \rangle) \\ \uparrow & & \uparrow \\ P^f & \xrightarrow{f} & B^3 \\ (f: \text{super-conformal}), & & \\ (P^f, ds_D^2) \hookrightarrow (D, ds_D^2), & (P^f, f^* ds_{B^3}^2) \xrightarrow{f} & (B^3, ds_{B^3}^2), \\ f^* ds_{B^3}^2 \leq 4(C^z)^2 ds_D^2. & & \end{array}$$

Figure 1: An analog of the Schwarz-Pick theorem.

Recalling the definition of a pole of a conformal map, the map $f := \psi\lambda$ with $\lambda = \lambda_0 + j\lambda_1$ for meromorphic functions λ_0 and λ_1 is a super-conformal map with poles. Hence, a super-conformal map with poles adopts properties of a meromorphic function.

The Weierstrass factorization theorem (see [7]) claims that, for a given divisor D , there exists a meromorphic function h with $(h) = D$. Because a

meromorphic function is a super-conformal map with left normal i , there exists a super-conformal map f with left normal i such that $(f) = D$. The map \bar{f} is a super-conformal map with left normal $-i$ such that $(\bar{f}) = D$.

Corollary 8. *For any divisor D on M and any anti-holomorphic map $N: M \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$, there exists a super-conformal map $f: M \setminus \text{supp } D \rightarrow \mathbb{H}$ with poles such that $(df)_{-N} = 0$ and $(f) = D$.*

Proof. By the Weierstrass factorization theorem, any divisor on M is a divisor of a meromorphic function. Let D be a divisor on M and λ a meromorphic function on M with divisor D . Then, $f := \psi\lambda: M \setminus \text{supp } D \rightarrow \mathbb{H}$ is a super-conformal map by Theorem 8 and $(f) = D$ by the definition of a divisor of a conformal map. \square

We assume that M is a connected open subset of a closed Riemann surface \tilde{M} . We denote by $C_1(M)$ the set of all one-chains in M . We define a map $\delta: C_1(M) \rightarrow \text{Div}(M)$ by, for $c: [0, 1] \rightarrow M$,

$$(\delta(c))(p) := \begin{cases} 1 & (p = c(1)), \\ -1 & (p = c(-1)), \\ 0 & (\text{otherwise}). \end{cases}$$

The following is an analog of the Abel-Jacobi theorem (see [7]).

Corollary 9. *Let D be a divisor on M with $\deg D = 0$. Then, D is the divisor of a super-conformal map from M with poles and left normal $N: M \rightarrow N(M) \subsetneq S^2$, if and only if there exists $c \in C_1(\tilde{M})$ such that $\delta(c) = D$ and*

$$\int_c \omega = 0$$

for every holomorphic one-form ω on \tilde{M} .

Proof. By the Abel-Jacobi theorem, the divisor D is a divisor of a meromorphic function λ on \tilde{M} . Hence $f := \psi\lambda: M \setminus \text{supp } D \rightarrow \mathbb{H}$ is a super-conformal map by Theorem 8. We see that $(f) = D$ by the definition of a divisor of a conformal map. \square

5 A conformal map with induced metric

We connect a conformal map with a classical surface.

Let $f: M \rightarrow \mathbb{H}$ be a conformal map with $(df)_{-N} = (df)^R = 0$. We induce a (singular) metric on M from \mathbb{H} by f . Consequently, the Gauss curvature K , the normal curvature K^\perp and the mean curvature vector \mathcal{H} of f can be defined. We have

$$df \bar{\mathcal{H}} = -N(dN)_N, \quad \bar{\mathcal{H}} df = R(dR)_R,$$

$$K|df|^2 = \frac{1}{2}(\langle *dR, RdR \rangle + \langle *dN, NdN \rangle),$$

$$K^\perp|df|^2 = \frac{1}{2}(\langle *dR, RdR \rangle - \langle *dN, NdN \rangle).$$

([2], Proposition 8, Proposition 9). A conformal map f is minimal if and only if N is holomorphic or, equivalently, R is holomorphic. Hence, if f is super-conformal and minimal, then N or R is a constant map. Then, f is a holomorphic map with respect to a complex structure of \mathbb{H} .

We consider the class of surfaces with $|K| = |K^\perp|$. We denote by σ the area element of the two sphere with radius one.

Lemma 6. *Let $f: M \rightarrow \mathbb{H}$ be a conformal map with $(df)_{-N} = (df)^R = 0$. If $|K| = |K^\perp|$, then $N^*\sigma = 0$ or $R^*\sigma = 0$.*

Proof. From the assumption, we have $K = \pm K^\perp$. Then $\langle *dN, NdN \rangle = 0$ or $\langle *dR, RdR \rangle = 0$. It is known that $\langle *dN, NdN \rangle = N^*\sigma$ and $\langle *dR, RdR \rangle = R^*\sigma$ ([2], Proposition 10). Hence the lemma holds. \square

If $N^*\sigma = 0$, then N is not anti-holomorphic. Hence, if f is super-conformal with $(df)_{-N} = (df)^R = 0$, then (a) $N^*\sigma = 0$ and R is anti-holomorphic or (b) $R^*\sigma = 0$ and N is anti-holomorphic.

Wintgen [18] showed that $K + |K^\perp| \leq |\mathcal{H}|^2$ for any conformal map and $K + |K^\perp| = |\mathcal{H}|^2$ if and only if a conformal map is super-conformal. A super-conformal map is called a Wintgen ideal surface in [16]. Chen [4] completely classified Wintgen ideal surfaces with $|K| = |K^\perp|$. We see that a Wintgen ideal surface with $|K| = |K^\perp|$ is a super-conformal map which is (i) minimal or (ii) $2K = 2|K^\perp| = |\mathcal{H}|^2$. If a super-conformal map with left normal N and right normal R is minimal, then $*dN = NdN = -NdN$ or $*dR = RdR = -RdR$. Hence N or R is constant.

Because $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, we can identify \mathbb{H} with \mathbb{C}^2 . A conformal map $f: M \rightarrow \mathbb{C}^2 \cong \mathbb{H}$ with $(df)_{-N} = (df)^R = 0$ is Lagrangian if and only if $R = je^{\theta i}$ where θ is a real-valued function ([12]). The map $R = je^{\theta i}$ is not anti-holomorphic. Hence if f is super-conformal and Lagrangian, then N is an anti-holomorphic map.

6 The Whitney sphere

We apply Theorem 2 to the Whitney sphere for an example of the previous sections.

Let (u, v) be the standard coordinate of \mathbb{R}^2 . The map $f: \mathbb{R}^2 \rightarrow \mathbb{H}$ defined by

$$f(u, v) = \frac{\sin u}{1 + (\cos u)^2} (\sin v - (\cos u \sin v)i - (\cos v)j - (\cos u \cos v)k)$$

is a parametrization of the Whitney sphere (see [4]). We define $\rho_{abc}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ by $\rho_{abc}(x_0, x_1, x_2, x_3) = (x_a, x_b, x_c)$. Figures 2, 3, 4, and 5 are plots of $(\rho_{abc} \circ f)(\mathbb{R}^2)$.

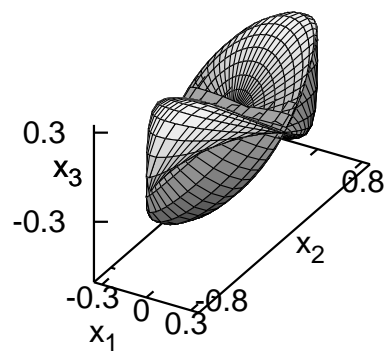


Figure 2: $(\rho_{123} \circ f)(\mathbb{R}^2)$

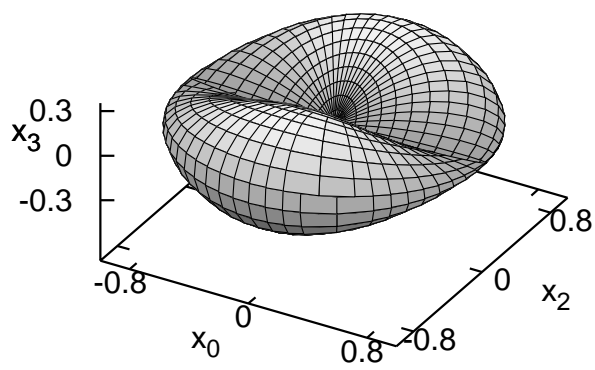


Figure 3: $(\rho_{023} \circ f)(\mathbb{R}^2)$

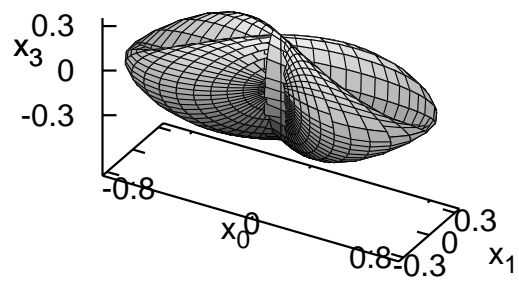


Figure 4: $(\rho_{013} \circ f)(\mathbb{R}^2)$

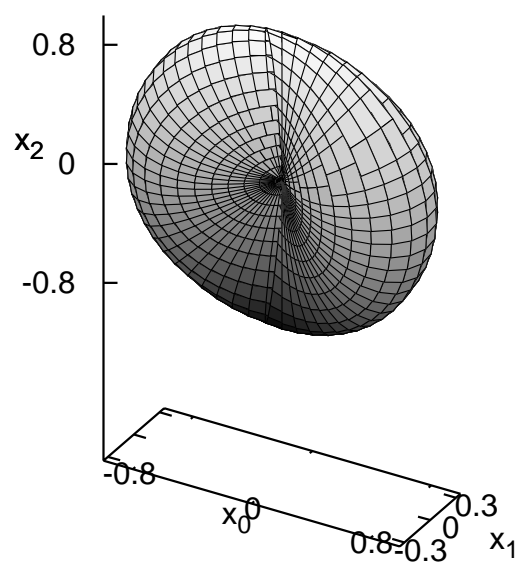


Figure 5: $(\rho_{012} \circ f)(\mathbb{R}^2)$

Castro [3] showed that the Whitney sphere is the only Lagrangian superconformal map from a closed Riemann surface. Chen [4] showed that the Whitney sphere is a Wintgen ideal surface with $|K| = |K^\perp|$. Because the Whitney sphere is closed, it is not a minimal surface. Hence $2K = 2|K^\perp| = |\mathcal{H}|^2$. By the discussion in the previous section, we can assume that the left normal of f is anti-holomorphic and the right normal is $je^{\beta i}$ for a real-valued function β .

We have that

$$\begin{aligned} df &= f_u du + f_v dv = \frac{f_u}{|f_u|} |f_u| du + \frac{f_v}{|f_v|} |f_v| dv, \\ f_u(u, v) &= \frac{1}{(1 + (\cos u)^2)^2} (\cos u \sin v ((\sin u)^2 + 2) + \sin v (3(\sin u)^2 - 2)i \\ &\quad + \cos u \cos v ((\cos u)^2 - 3)j + \cos v (3(\sin u)^2 - 2)k), \\ f_v(u, v) &= \frac{\sin u}{1 + (\cos u)^2} (\cos v + (-\cos u \cos v)i + (\sin v)j + (\cos u \sin v)k), \\ |f_u|^2 &= \frac{1}{1 + (\cos u)^2}, \quad \langle f_u, f_v \rangle = 0, \quad |f_v|^2 = \frac{(\sin u)^2}{1 + (\cos u)^2}. \end{aligned}$$

Hence f is not an immersion at $(n\pi, v)$ for any integer n and any $v \in \mathbb{R}$. Define a complex structure J of \mathbb{R}^2 by $J(f_u/|f_u|) = f_v/|f_v|$. We have

$$*df = \frac{f_v}{|f_v|} |f_u| du - \frac{f_u}{|f_u|} |f_v| dv.$$

Then the left normal N and the right normal R are respectively

$$\begin{aligned} N(u, v) &= \frac{f_v(u, v)}{|f_v(u, v)|} \left(\frac{f_u(u, v)}{|f_u(u, v)|} \right)^{-1} \\ &= \frac{\sin u}{|\sin u|(1 + (\cos u)^2)} ((-2(\sin u)^2 \cos v \sin v)i \\ &\quad + (2 \cos v)j + ((\sin u)^2 (2(\sin u)^2 - 1)k), \\ R(u, v) &= \left(\frac{f_v(u, v)}{|f_v(u, v)|} \right)^{-1} \frac{f_u(u, v)}{|f_u(u, v)|} \\ &= \frac{\sin u}{|\sin u|(1 + (\cos u)^2)^2} \left((-4 \cos u \sin u)j + \frac{(\sin u)^4 + 4(\sin u)^2 - 4}{\sin u} k \right) \end{aligned}$$

Let $V := \{(u, v) \in \mathbb{R}^2 \mid N(u, v) = -i\}$ and define a map $\lambda: \mathbb{R}^2 \setminus V \rightarrow \mathbb{H}$ by $f = (N + i)\lambda$. Then,

$$\begin{aligned} \lambda(u, v) &= \frac{\cos u \sin u (\sin v + \sin u \cos v)}{D} + \frac{\sin u (\sin v - \sin u \cos v)}{D} i \\ &\quad + \frac{\cos u \sin v (\sin u \sin v + \cos v)}{D} j + \frac{\sin u (\sin u \sin v - \cos v)}{D} k, \\ D &= 4(\sin u)^3 \cos v \sin v + (\sin u)^4 - (\sin u)^2 - 2 \end{aligned}$$

satisfies the equation $(d\lambda)_{-i} = 0$ with respect to J . Since f has no zero on $\mathbb{R}^2 \setminus V$, the map λ has no zero on $\mathbb{R}^2 \setminus V$.

7 Minimal surfaces

We give a factorization of a minimal surface.

Let $f: M \rightarrow \mathbb{H}$ be a minimal surface with $(df)_{-N} = (df)^R = 0$. A minimal surface $g: M \rightarrow \mathbb{H}$ such that $dg = - * df$ is called a conjugate minimal surface of f . There exists a conjugate minimal surface of f if and only if $*df$ is exact. A conjugate minimal surface g shares the same left normal and the same right normal with the original minimal surface f . If there exists a conjugate minimal surface $g: M \rightarrow \mathbb{H}$, then the holomorphic map $f + ig: M \rightarrow \mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}^4$ is called a holomorphic null curve.

For a factorization of a minimal surface, we assume that M is simply connected and induce a map μ as follows.

Let $N: M \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$ be a holomorphic map. By Lemma 1, there exists $a \in \mathbb{H}$ with $|a| = 1$ such that $\psi := -Na + ai$ does not vanish on M . Let λ_0 and λ_1 be holomorphic functions on M and $\lambda := \lambda_0 + \lambda_1 j$. Then, $(d\lambda)_{-i} = 0$. Put $Q_{\lambda_0, \lambda_1} := \{p \in M \mid (dN)_p \lambda(p) = 0\}$. Because $(\psi d\lambda)_N = 0$ and $(dN a\lambda)_N = 0$, the equation $\psi d\lambda = dN a\lambda \mu$ defines a map $\mu: M \setminus Q_{\lambda_0, \lambda_1} \rightarrow \mathbb{H}$.

If $N: M \rightarrow N(M) \subsetneq S^2 \cong \mathbb{CP}^1$ is holomorphic, then $-N$ is anti-holomorphic by Corollary 2. The map $\psi\lambda$ with $(d\lambda)_{-i} = 0$ is super-conformal with left normal $-N$ by Theorem 2. Because N is holomorphic and $(d\lambda)_{-i} = 0$, the set Q_{λ_0, λ_1} is discrete.

Proof of Theorem 3. (a) We assume that $\Phi := f + ig: M \rightarrow \mathbb{C} \otimes \mathbb{H}$ is a holomorphic null curve such that f and $g: M \rightarrow \mathbb{H}$ are minimal surfaces with left normal N . We have $d(dN f) = -dN \wedge df = -(dN)^N \wedge (df)_N = 0$. Then, the one-form $dN f$ on M is exact. Hence, there exists a function $\Lambda: M \rightarrow \mathbb{H}$ such that $dN f = d\Lambda$. We define $\lambda: M \rightarrow \mathbb{H}$ by $\lambda := \psi^{-1}\Lambda$. Then,

$$dN f = -dN a\lambda + \psi d\lambda = d(\psi\lambda).$$

Because $(dN f)_N = 0$ and $(dN a\lambda)_N = 0$, we have $(\psi d\lambda)_N = 0$. Then,

$$(\psi d\lambda) - N * (\psi d\lambda) = \psi(d\lambda + i * d\lambda) = 0.$$

Hence, $(d\lambda)_{-i} = 0$. Then,

$$dN f = dN a\lambda(\mu - 1)$$

on $M \setminus Q_{\lambda_0, \lambda_1}$. Then, $f = a\lambda(\mu - 1)$. Because the left hand side is defined on M , the right hand side is extended to M . Then,

$$\begin{aligned} - * df &= -N df = -d(Nf) + dN f = -d(Nf) - dN a\lambda + (-Na + ai) d\lambda \\ &= d(-Na\lambda(\mu - 1) + (-Na + ai)\lambda) = d(-Na\lambda\mu + ai\lambda). \end{aligned}$$

Hence $g = -Na\lambda\mu + ai\lambda$ up to an additive constant.

(b) We have

$$dN f = dN (a\lambda(\mu - 1)) = -dN a\lambda + dN a\lambda\mu$$

$$= -dN a\lambda + \psi d\lambda = d(\psi\lambda).$$

Differentiating the above equation, we have

$$-dN \wedge df = (dN)^N \wedge (df)_{-N} = 0.$$

Hence $(df)_{-N} = 0$. Then f is a minimal surface with left normal N . Then,

$$\begin{aligned} - * df &= -N df = -d(Nf) + dN f = -d(Nf) - dN a\lambda + (-Na + ai) d\lambda \\ &= d(-Na\lambda(\mu - 1) + (-Na + ai)\lambda) = d(-Na\lambda\mu + ai\lambda) \end{aligned}$$

Hence, g is a minimal surface with left normal N and $\Phi := f + ig$ is a holomorphic null curve. \square

The equation $f = a\lambda(\mu - 1)$ is a factorization of a minimal surface which has a conjugate minimal surface g and $\Phi := f + gi$ is a holomorphic null curve. The arrangement of the zeros of f is unclear because that of $\mu - 1$ is unclear. However, we have the following property.

Corollary 10. *Let $f = a\lambda(\mu - 1): M \rightarrow \mathbb{H}$ be a minimal surface factored by Theorem 3. A point on M is a branch point of a super-conformal map $\psi\lambda$ if and only if it is a zero of f or a branch point of N .*

Proof. From the proof of Theorem 3, we have $dN f = d(\psi\lambda)$. Hence the corollary holds. \square

8 The Enneper surface

We apply Corollary 10 to the Enneper surface for an example of the previous section.

Let z be the standard holomorphic coordinate of \mathbb{C} and (x, y) the real coordinate such that $z = x + iy$. The map $f: \mathbb{C} \rightarrow \text{Im } \mathbb{H}$ defined by

$$f(z) = \left(x - \frac{1}{3}x^3 + xy^2\right)i + \left(-y - x^2y + \frac{1}{3}y^3\right)j + (x^2 - y^2)k$$

is a parametrization of the Enneper surface and $g: \mathbb{C} \rightarrow \text{Im } \mathbb{H}$ defined by

$$g(z) = \left(y - x^2y + \frac{1}{3}y^3\right)i + \left(x - xy^2 + \frac{1}{3}x^3\right)j + (2xy)k$$

is its conjugate surface. These surface is a complete minimal surface total curvature -4π . Hence these are conformal maps with poles by Proposition 1. The map f has only one zero at the origin. The left normal is

$$N(z) = \frac{2x}{x^2 + y^2 + 1}i + \frac{2y}{x^2 + y^2 + 1}j + \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}k.$$

The stereographic projection of N from k is z . Hence, N is unbranched. Let $a := (i + k)/\sqrt{2}$, then $-Na + ai$ does not vanishes on \mathbb{C} .

Differentiating N , we have

$$dN = \left(\frac{2(y^2 - x^2 + 1)}{(x^2 + y^2 + 1)^2} i + \frac{-4xy}{(x^2 + y^2 + 1)^2} j + \frac{4x}{(x^2 + y^2 + 1)^2} k \right) dx \\ + \left(\frac{-4xy}{(x^2 + y^2 + 1)^2} i + \frac{2(x^2 - y^2 + 1)}{(x^2 + y^2 + 1)^2} j + \frac{4y}{(x^2 + y^2 + 1)^2} k \right) dy.$$

Then,

$$dN f = \left[-\frac{2x(y^4 + 2x^2y^2 + 6y^2 + x^4 + 2x^2 + 3)}{3(x^2 + y^2 + 1)^2} \right. \\ + \frac{4xy(2y^2 + 3)}{3(x^2 + y^2 + 1)^2} i \\ + \frac{2(3y^4 + 3y^2 + x^4 + 3x^2)}{3(x^2 + y^2 + 1)^2} j \\ + \left. \frac{2(y^5 + 2x^2y^3 - 2y^3 + x^4y + 6x^2y - 3y)}{3(x^2 + y^2 + 1)^2} k \right] dx \\ + \left[\frac{2(y^5 + 2x^2y^3 + 2y^3 + x^4y + 6x^2y + 3y)}{3(x^2 + y^2 + 1)^2} \right. \\ + \frac{2(y^4 + 3y^2 + 3x^4 + 3x^2)}{3(x^2 + y^2 + 1)^2} i \\ + \frac{4xy(2x^2 + 3)}{3(x^2 + y^2 + 1)^2} j \\ + \left. \frac{2x(y^4 + 2x^2y^2 + 6y^2 + x^4 - 2x^2 - 3)}{3(x^2 + y^2 + 1)^2} k \right] dy \\ = d \left(\frac{y^4 + 5y^2 - x^4 - x^2 + 2}{3(x^2 + y^2 + 1)} + \frac{2y(y^2 + 3x^2)}{3(x^2 + y^2 + 1)} i \right. \\ + \frac{2x(3y^2 + x^2)}{3(x^2 + y^2 + 1)} j + \left. \frac{2x(y^3 + x^2y - 3y)}{3(x^2 + y^2 + 1)} k \right).$$

Hence

$$\psi(z)\lambda(z) := \frac{y^4 + 5y^2 - x^4 - x^2 + 2}{3(x^2 + y^2 + 1)} + \frac{2y(y^2 + 3x^2)}{3(x^2 + y^2 + 1)} i \\ + \frac{2x(3y^2 + x^2)}{3(x^2 + y^2 + 1)} j + \frac{2x(y^3 + x^2y - 3y)}{3(x^2 + y^2 + 1)} k$$

is a super-conformal map with left normal $-N$. By Corollary 10, the map $\psi\lambda$ has only a single branch point at the origin.

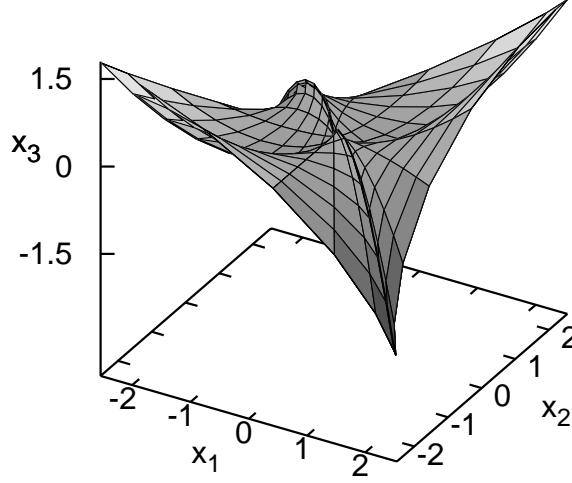


Figure 6: $(\rho_{123} \circ \psi \lambda)(U)$

We fix a set $U := \{(x, y) \mid 0 \leq x^2 + y^2 \leq 9\} \subset \mathbb{R}^2$. We define $\rho_{abc}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ by $\rho_{abc}(x_0, x_1, x_2, x_3) = (x_a, x_b, x_c)$. Figures 6, 7, 8, and 9 are plots of $(\rho_{abc} \circ \psi \lambda)(U)$. We have

$$\begin{aligned}
& \lambda(z) = (-N(z)a + ai)^{-1} \\
& \times \left(\frac{y^4 + 5y^2 - x^4 - x^2 + 2}{3(x^2 + y^2 + 1)} + \frac{2y(y^2 + 3x^2)}{3(x^2 + y^2 + 1)}i \right. \\
& \left. + \frac{2x(3y^2 + x^2)}{3(x^2 + y^2 + 1)}j + \frac{2x(y^3 + x^2y - 3y)}{3(x^2 + y^2 + 1)}k \right) \\
& = \frac{3xy^2 - 3y^2 - x^3 + 3x^2 + 2x - 2}{6\sqrt{2}} \\
& \quad + \frac{y(y^2 - 3x^2 + 6x + 2)}{6\sqrt{2}}i \\
& \quad - \frac{3xy^2 + 3y^2 - x^3 - 3x^2 + 2x + 2}{6\sqrt{2}}j
\end{aligned}$$

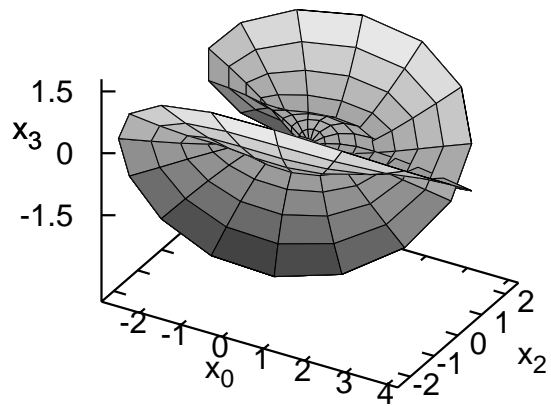


Figure 7: $(\rho_{023} \circ \psi\lambda)(U)$

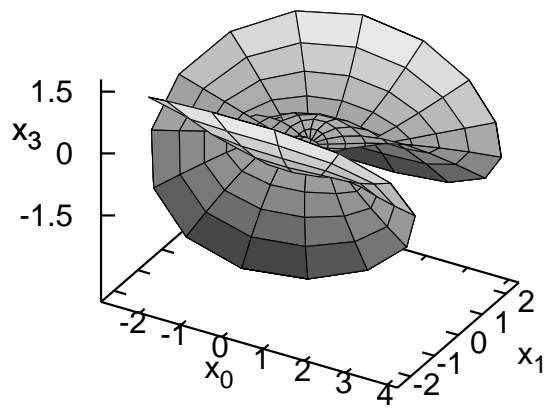


Figure 8: $(\rho_{013} \circ \psi\lambda)(U)$

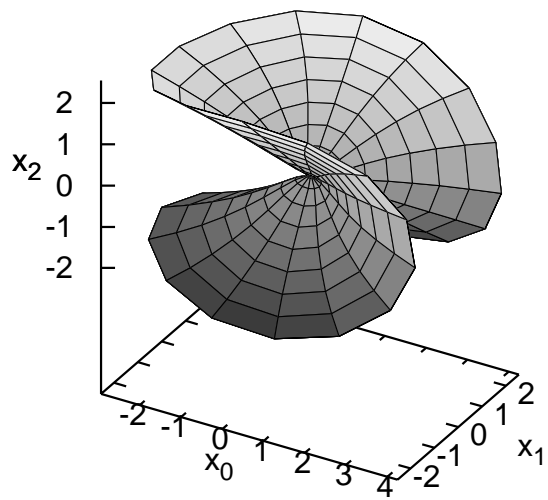


Figure 9: $(\rho_{012} \circ \psi\lambda)(U)$

$$-\frac{y(y^2 - 3x^2 - 6x + 2)}{6\sqrt{2}}k.$$

The map μ with $(-Na + ai) d\lambda = dN a\lambda\mu$ becomes

$$\begin{aligned}\mu(z) = & \frac{3y^6 + 9x^2y^4 + 29y^4 + 9x^4y^2 + 18x^2y^2 + 16y^2 + 3x^6 + 13x^4 - 8x^2 + 4}{D} \\ & + \frac{-2y(18x^2y^2 - 13y^2 - 6x^4 + 27x^2 - 6)}{D}_i \\ & + \frac{-2x(6y^4 - 18x^2y^2 - 27y^2 + 5x^2 - 6)}{D}_j \\ & + \frac{4xy(17y^2 - 13x^2 + 6)}{D}k, \\ D = & y^6 + 3x^2y^4 + 13y^4 + 3x^4y^2 + 18x^2y^2 + 16y^2 + x^6 + 5x^4 - 8x^2 + 4.\end{aligned}$$

Hence,

$$\begin{aligned}\mu(z) - 1 = & \frac{2(y^6 + 3x^2y^4 + 8y^4 + 3x^4y^2 + x^6 + 4x^4)}{D} \\ & + \frac{-2y(18x^2y^2 - 13y^2 - 6x^4 + 27x^2 - 6)}{D}_i \\ & + \frac{-2x(6y^4 - 18x^2y^2 - 27y^2 + 5x^2 - 6)}{D}_j \\ & + \frac{4xy(17y^2 - 13x^2 + 6)}{D}k.\end{aligned}$$

Then $f = a\lambda(\mu - 1)$ by Theorem 3.

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